Renmin Summer Course Problems

① Suppose there is a $u \in C^2(\bar{\Omega})$, where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with smooth boundary, so that

$$\Delta u = f \quad \text{in } \Omega \quad \ldots$$

$$\nabla u \cdot \vec{\eta} = g \quad \text{on } \partial\Omega \quad \ldots$$

where $f \in C(\bar{\Omega})$ and $g \in C(\partial\Omega)$. Prove that

$$\int_{\Omega} f \, dx = \int_{\partial\Omega} g \, dS.$$

② Suppose we think of an individual organism that moves along a line in discrete time steps by jumping one spatial step to the right with probability $1/2$ or one step to the left with probability $1/2$. Let Δt denote the time step and Δx denote the space step, and let $p(x,t)$ denote the probality that the individual is at location x at time t.

ⓐ Explain why

$$(*) \qquad p(x, t+\Delta t) = (1/2)\, p(x-\Delta x, t) + (1/2)\, p(x+\Delta x, t)$$

ⓑ Use Taylor expansion to obtain from $(*)$ that

(**) $\dfrac{\partial p}{\partial t}(x,t)\,\Delta t + \text{h.o.t.}\,(\Delta t)$

$$= \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x,t)\,(\Delta x)^2 + \text{h.o.t}\,(\Delta x)$$

(c) Assuming that $\Delta x \to 0$, $\Delta t \to 0$ in such a way that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{2\,\Delta t} = d,$$

show that $\dfrac{\partial p}{\partial t}(x,t) = d\,\dfrac{\partial^2 p}{\partial x^2}(x,t)$.

3 Complete the following derivation that the solution to the problem

$$(*) \qquad \frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

on $-\infty < x < \infty$, $0 < t < \infty$

subject to the initial condition $u(x,0) = \phi(x)$

is given by $u(x,t) = \dfrac{1}{\sqrt{4\pi k t}} \displaystyle\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\,\phi(y)\,dy.$ (**)

(Here we assume ϕ is bounded and continuous on $\mathbb{R}$.)

ⓐ Our derivation relies on some invariance properties of solutions to (*). Verify them as indicated.

1. If u solves (*) and $y \in \mathbb{R}$ is given, $v^y(x,t) = u(x-y,t)$ solves (*)
2. If u solves (*), so does any derivative of u. In particular, show that $\frac{\partial u}{\partial x}$ solves (*).
3. A linear combination of solutions is a solution.
4. Integrals of solutions of (*) are solutions. In particular, suppose $S(x,t)$ solves (*) and

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, g(y)\, dy$$

where $g(y)$ is bounded and continuous. Assuming the convergence of v at $\pm\infty$ justifies exchanging order of differentiation and integration show that v satisfies (*).
5. If u solves (*), $a > 0$ and $v(x,t) = u(\sqrt{a}\,x, at)$, then v solves (*).

ⓑ Solve the problem when ϕ is the bounded (but not continuous) function $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$

1. Suppose $Q(x,t)$ is a solution. Show $Q(\sqrt{a}\,x, 0) = Q(x,0)$ for any $a > 0$. Conclude that $Q(x,t) = Q(\sqrt{a}\,x, at)$ for any $a > 0$.
2. Fix a $t_0 > 0$. Show that $Q(x,t_0) = Q\left(\frac{x}{\sqrt{t_0}}, 1\right)$
Conclude that $Q(x,t) = Q\left(\frac{x}{\sqrt{t}}, 1\right)$

3. So $Q(x,t)$ is a function of $x/\sqrt{t}$. Look for $Q(x,t) = g(p)$, with $p = \dfrac{x}{\sqrt{4kt}}$.

Show that if $' = \dfrac{d}{dp}$, $g''(p) + 2p\,g'(p) = 0$

4. Use the definition of H and the fact that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \text{ to show that}$$

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp$$

(c) Let $S = \dfrac{\partial Q}{\partial x}$. Define

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$$

Show that u is given by (**)
Establish that

$$\int_{-\infty}^{\infty} S(x-y,t)\,|\phi(y)|\,dy$$

is convergent for any $\phi(y)$ which is bounded and continuous. So u makes sense as a solution of (*) for any such ϕ.

(d) Establish that $u(x_0,0) = \lim\limits_{\substack{x \to x_0 \\ t \to 0^+}} u(x,t) = \phi(x_0)$

1. Calculate that for any $x \in \mathbb{R}$, $t > 0$ $\displaystyle\int_{-\infty}^{\infty} S(x-y,t)\,dy = 1$.

2. Show that for any $\delta > 0$, $x \in \mathbb{R}$, $t > 0$

$$u(x,t) - \phi(x_0)$$

$$= \int_{|y-x_0|<\delta} S(x-y,t)\,[\phi(y) - \phi(x_0)]\,dy$$

$$+ \int_{|y-x_0|\geq\delta} S(x-y,t)\,[\phi(y) - \phi(x_0)]\,dy$$

3. Let $\varepsilon > 0$ be given. Show that if $\delta > 0$ is such that $|\phi(y) - \phi(x_0)| < \varepsilon/2$ if $|y-x_0| < \delta$ then the absolute value of the first integral in (C2) is less than $\varepsilon/2$.

4. Let M be a bound on $|\phi(x)|$. Show that the absolute value of the second integral is bounded by

$$2M\left[\frac{1}{\sqrt{\pi}} \int_{\frac{x-x_0+\delta}{\sqrt{4kt}}}^{\infty} e^{-w^2}\,dw + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x-x_0-\delta}{\sqrt{4kt}}} e^{-w^2}\,dw\right] \quad (\#)$$

5. Suppose that $|x-x_0| < \delta/2$. Choose $c > 0$ so that

$$\int_c^{\infty} e^{-w^2}\,dw < \varepsilon/8M \quad \text{and} \quad \int_{-\infty}^{-c} e^{-w^2}\,dw < \varepsilon/8M$$

Show that if $|x-x_0| < \delta/2$ and $t < \frac{\delta^2}{16kc^2}$

then (#) is less than $\varepsilon/2$.

6. Conclude that $\displaystyle\lim_{\substack{x \to x_0 \\ t \to 0^+}} u(x,t) = \phi(x_0)$

(4) Suppose $u, v \in C^2(\bar{\Omega})$ with $u, v > 0$ on $\bar{\Omega}$ satisfy

$$-\Delta u = \lambda u \quad \text{in } \Omega$$

$$\alpha \nabla u \cdot \eta + (1-\alpha) u = 0 \quad \text{on } \partial\Omega$$

and

$$-\Delta v = \gamma v \quad \text{in } \Omega$$

$$\beta \nabla u \cdot \eta + (1-\beta)$$

with $\alpha, \beta \in (0,1)$.

(a) Prove that $\lambda > 0$ and $\gamma > 0$.

(b) Prove that $\alpha < \beta \Rightarrow \lambda < \gamma$.

(5) (a) Prove that if X is a finite dimensional normed linear space, all norms are equivalent.

(b) Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\mathbb{R}^N$ and $d_1(x,y) = \|x-y\|_1$ and $d_2(x,y) = \|x-y\|_2$ are the corresponding induced metrics. Let $B_i(x,\varepsilon) = \{y \in \mathbb{R}^N \mid d_i(x,y) < \varepsilon\}$. Show that $B_i(x,\varepsilon)$ is an open set in the topology determined by d_j. Here $i,j \in \{1,2\}$ and $i \neq j$.

(6) Suppose that y solves the initial value problem

$$(*) \qquad y' = y f(y) \qquad y(0) = y_0$$

where f is Lipschitz continuous. Show that:

(a) If $y_0 > 0$, then y is positive on its maximal interval of existence.

(b) If there is an $M_0 > 0$ so that $f(y) < 0$ for $y > M_0$, then any positive solution to $(*)$ exists globally!

(7) Use the theory of upper and lower solutions for parabolic equations to establish the following:

Suppose that $f(x,u)$ and $\frac{\partial f}{\partial u}(x,u)$ are Hölder continuous in x and continuous in u for $(x,u) \in \bar{\Omega} \times \mathbb{R}$. Suppose that $\underline{u} \in C^2(\bar{\Omega})$ satisfies

$$-\Delta \underline{u} \leq f(x, \underline{u})$$

on $\bar{\Omega}$ with $\underline{u} \leq 0$ on $\partial\Omega$. Suppose that $v = v(x,t)$ $\in C^{2,1}(\Omega \times [0,T])$ satisfies

$$\frac{\partial v}{\partial t} = \Delta v + f(x,v) \qquad \text{in } \Omega \times (0,T]$$

$$v = 0 \qquad \text{on } \partial\Omega \times (0,T]$$

$$v(x,0) = \underline{u}(x) \qquad \text{in } \Omega$$

Then either $v(x,t) \equiv \underline{u}(x)$ or $v(x,t)$ is increasing in $t \in [0,T]$.

(8) Consider (3.1):

$$u_t = d u_{xx} + r u \qquad \text{on } (0,1) \times (0,\infty).$$

$$(*) \qquad u(0,t) = u(1,t) = 0 \qquad \text{on } (0,\infty)$$

$$u(x,0) = f(x)$$

Let $b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$ be the n^{th} Fourier sine coefficient of the initial configuration $f(x)$. Assume that $\{b_n\}_{n\geq 1}^{\infty}$ is bounded. Fill in the remaining details to establish that

$$u(x,t) = \sum_{n\geq 1}^{\infty} b_n e^{(r - d n^2\pi^2)t} \sin(n\pi x)$$

is a classical solution to $(*)$ for all $t > 0$.

(9) Let Ω be a bounded open domain in $\mathbb{R}^n$. Let $C^0(\bar{\Omega})$ and $C^\alpha(\bar{\Omega})$ denote the spaces of continuous real valued

functions and Hölder continuous real valued functions on $\bar{\Omega}$ with the usual norms. Let $E : C^{\alpha}(\bar{\Omega}) \to C^{0}(\bar{\Omega})$ denote the operator E which embeds $C^{\alpha}(\bar{\Omega})$ into $C^{0}(\bar{\Omega})$ given by

$$Eu = u.$$

Prove that E is a compact linear operator with norm 1.

(10) Let X be a Banach space and let $K : X \to X$ be compact. Complete the details of the following.

(a) If $R(K)$, the range of K, is closed, then $\dim R(K) < \infty$.
(Hint: Open Mapping Theorem)

(b) $\dim N(I-K) < \infty$
(Hint: $K(N(I-K)) = N(I-K)$.)

(c) $R(I-K)$ is closed.

1. Show that $X = N(I-K) \oplus M$, where M is a closed subspace of X.
2. Define $S : M \to X$ by $Sx = x - Kx$. S is 1-1 and $R(S) = R(I-K)$. Show that there is an $r > 0$ so that

$$r\|x\| \le \|Sx\|$$

for all $x \in M$. Conclude that $R(S)$ is closed.

(d) Since $R(I-K)$ is closed, it is known that $R(I-K) = {}^{\circ}(N(I-K'))$, the annihilator of the null space of the adjoint of $I-K$. It is also known that $\dim N((I-K')) = \dim N((I-K))$. Take these pieces of information as given. Show that $R(I-K) = X$ $\iff N((I-K)) = \{0\}$.

(11) Complete the proof of Theorem 4.9 in the notes; i.e. show how to realize the mapping from f to u in the problem

$$\begin{aligned} Lu &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

as a compact map from $C^{1+\alpha}(\bar{\Omega})$ to $C^{1+\alpha}(\bar{\Omega})$.

(12) Let X be a Banach space and assume that $K \in B(X)$ is compact. Prove that for any $\varepsilon > 0$, there are at most finitely many values of $\lambda \in \mathbb{C}$ with $|\lambda| \geq \varepsilon$ so that

$$Ku = \lambda u$$

has a nontrivial solution.

(13) Fill in the details in the outline below showing that if $A \in B(X)$, and $u_0 \in X$, then

$$e^{tA} u_0$$

solves

$$(*) \qquad \frac{du}{dt} = Au, \quad t > 0$$

$$u(0) = u_0$$

where $e^{tA} \in B(X)$ is the operator given by $\sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k$, which converges for any $A \in B(X)$.

(a) Establish that $C_n = \sum_{k=0}^{n} \frac{1}{k!} t^k A^k$ is a Cauchy sequence

sequence in $B(X)$. Hence e^{tA} as defined exists in $B(X)$.
Moreover, $\|e^{tA}\| \leq e^{|t|\,\|A\|}$

(b) Show that if $B, C \in B(X)$ and $BC = CB$, then $e^{B+C} = e^B e^C$.

(c) Set $u(t) = e^{tA} u_0$. Show that

$$\frac{u(t+h) - u(t)}{h} = \left[\frac{e^{hA} - I}{h}\right] u(t)$$

Hence $$\frac{u(t+h) - u(t)}{h} - Au(t)$$

$$= \left[\frac{e^{hA} - I}{h} - A\right] u(t)$$

and $$\left\| \frac{e^{hA} - I}{h} - A \right\| \leq \frac{e^{|h|\,\|A\|} - 1}{|h|} - \|A\| \to 0$$

Hence $$u'(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} = Au(t)$$

(14) Fill in the details below to show that the solution of (*) in #13 is unique.

(a) Suppose (*) admits two solutions u_1, u_2. Then $u = u_1 - u_2$ solves

(**) $\quad \frac{du}{dt} = Au$ for $t > 0$ with $u(0) = 0$

Set $v(t) = e^{-tA} u(t)$. Calculate that

$$\frac{v(t+h) - v(t)}{h} = e^{-(t+h)A} \left[\frac{u(t+h) - u(t)}{h} \right]$$

$$+ \left[\frac{e^{-hA} - I}{h} \right] v(t)$$

(b) $\lim_{h \to 0} e^{-(t+h)A} = e^{-tA}$ and $\lim_{h \to 0} \left[\frac{e^{-hA} - I}{h} \right] = -A$

(c) Conclude from (a) and (b) that

$$\frac{d}{dt} \left(e^{-tA} u(t) \right) = e^{-tA} \left(\frac{du}{dt} - Au(t) \right) = 0$$

(d) Let $f \in X'$, the dual space of X.
Set $F(t) = f(e^{-tA} u(t))$.
Show that

$$\frac{F(t+h) - F(t)}{h} = f\left(\frac{e^{-(t+h)A} u(t+h) - e^{-tA} u(t)}{h} \right)$$

$$\to f\left(e^{-tA} \left(\frac{du}{dt} - Au(t) \right) \right) = f(0) = 0 \quad \text{as } h \to 0$$

(e) Conclude that $F: \mathbb{R} \to \mathbb{R}$ is differentiable for $t > 0$, $F'(t) = 0$, F is continuous for $t \geq 0$ with $F(0) = 0$.

(f) So $f(e^{-tA}u(t)) \equiv 0$ for $f \in X'$. Conclude that $u(t) = 0$ in X for $t \geq 0$.

The next set of exercises concern the following extension of the result of problem #13. Suppose that A is a closed linear operator in X with $D(A)$ a dense subspace of X.

A scalar λ is said to be in the resolvent set of A, denoted $\rho(A)$, if $N(A-\lambda) = 0$ and $R(A-\lambda) = X$, so that $(A-\lambda I)^{-1} \in B(X)$.

Assume that $[b, \infty) \subseteq \rho(A)$ where $b \geq 0$ and that there is a constant a so that

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{a+\lambda} \quad \text{for } \lambda \geq b$$

Then there is a family $\{E_t\}$ of operators in $B(X)$, $t \geq 0$ so that

(a) $E_s E_t = E_{s+t}$, $s \geq 0$, $t \geq 0$

(b) $E_0 = I$

(c) $\|E_t\| \leq e^{-at}$, $t \geq 0$

(d) $E_t x$ is continuous on $t \geq 0$ for each $x \in X$

(e) $E_t x$ is differentiable in $t \geq 0$ for each $x \in D(A)$

with $\dfrac{d E_t x}{dt} = A E_t x$

(f) $E_t (\lambda - A)^{-1} = (\lambda - A)^{-1} E_t$, $\lambda \geq b$, $t \geq 0$

(15) Assuming (a)-(f), let $u_0 \in D(A)$ and set $u(t) = E_t u_0$.
Show that $\dfrac{du(t)}{dt} = Au(t)$ for $t > 0$

$$u(0) = u_0.$$

(16) Establish the following result. Let D be dense in Banach space X and let $\{B_\lambda\}$ be a family of operators in $B(X)$ so that $\|B_\lambda\| \leq M$ for $\lambda \geq K$. If $B_\lambda x$ converges as $\lambda \to \infty$ for each $x \in D$, then there is a $B \in B(X)$ $\ni$ $\|B\| \leq M$ and $B_\lambda x \to Bx$ for all $x \in D$.

(17) Assume the result holds if $a > 0$. Suppose that $a \leq 0$ and let $B = A + (a-1)I$.

(a) Show $\lambda \in \rho(B) \Leftrightarrow \lambda - a + 1 \in \rho(A)$

(b) Show that there is a $b_1 > 0$ so that $[b_1, \infty) \subseteq \rho(B)$ and for $\lambda \geq b_1$, $\lambda - a + 1 \geq b$

and $\|(\lambda - B)^{-1}\| \leq \dfrac{1}{1+\lambda}$

(c) Apply the result for $a = 1$ to get the corresponding family $\{E_t\}_{t \geq 0}$ for B.

Set $F_t = e^{t-at} E_t$, $t \geq 0$. Show that (a)-(f) hold for $\{F_t\}$ relative to A.

(18) Now assume $a > 0$. For $\lambda \geq b$, set $A_\lambda = \lambda A(\lambda - A)^{-1}$.

(a) Since $(\lambda - A)^{-1}$ maps X to $D(A)$, derive that

$$\lambda A(\lambda - A)^{-1} = -\lambda I + \lambda^2 (\lambda - A)^{-1} \in B(X)$$

(b) Show that $e^{tA_\lambda} = e^{-t\lambda} e^{t\lambda^2(\lambda - A)^{-1}}$
and that

$$\| e^{tA_\lambda} \| \leq e^{\frac{-at\lambda}{a+\lambda}}, \quad t \geq 0, \ \lambda \geq b$$

(c) Set $B_\lambda = A(\lambda - A)^{-1}$ for $\lambda \geq B$.

Show that $\| B_\lambda \| \leq 1 + \frac{\lambda}{a+\lambda} \leq 2$

and that $B_\lambda x \to 0$ for $x \in D(A)$ as $\lambda \to \infty$
Conclude from #16 that
$\lambda(\lambda - A)^{-1} x \to x$ for all $x \in X$ as $\lambda \to \infty$
So $\lambda(\lambda - A)^{-1} Ax \to Ax$ for all $x \in D(A)$
as $\lambda \to \infty$.
Conclude that $A_\lambda x \to Ax$ for $x \in D(A)$, as $\lambda \to \infty$.

(d) Establish that $\lim_{\lambda \to \infty} e^{tA_\lambda} x$ exists for all $x \in X$,
$t \geq 0$.

1. For $\lambda \geq b$, $A_\lambda = -\lambda + \lambda^2(\lambda - A)^{-1}$. Show that $(\lambda - A)^{-1}(\mu - A)^{-1} = (\mu - A)^{-1}(\lambda - A)^{-1}$ and hence for $\lambda, \mu \geq b$ $A_\lambda A_\mu = A_\mu A_\lambda$.

2. Set $V_s = \exp[stA_\lambda + (1-s)tA_\mu]$, $s \in [0,1]$

and let $v(s) = V_s x$, $x \in X$

Show that $v(s) = \exp[st(A_\lambda - A_\mu)]\exp[tA_\mu]x$

3. Calculate that

$$v'(s) = t(A_\lambda - A_\mu)v(s)$$

4. Show that

$$[\exp(tA_\lambda) - \exp(tA_\mu)](x)$$

$$= v(1) - v(0)$$

$$= \int_0^1 v'(s)\,ds = t\int_0^1 V_s(A_\lambda - A_\mu)x\,ds$$

5. Show that

$$V(s) = e^{(-st\lambda - (1-s)t\mu)}\exp[st\lambda^2(\lambda - A)^{-1} + (1-s)t\mu^2(\mu - A)^{-1}]$$

and that $\|V(s)\| \leq \exp\left[\frac{-ast\lambda}{a+\lambda} + \left(\frac{-a(1-s)t\mu}{a+\mu}\right)\right] \leq 1$

6. Conclude from (4) and (5) that

$$\|(e^{tA_\lambda} - e^{tA_\mu})x\| \leq t\int_0^1 ds\, \|(A_\lambda - A_\mu)x\|$$

for $x \in X$

7. If $x \in D(A)$, $A_\lambda x \to Ax$. Use (6) to conclude if $x \in D(A)$, $\|e^{tA_\lambda}x - e^{tA_\mu}x\| \to 0$ as $\lambda, \mu \to \infty$. Conclude $\lim_{\lambda\to\infty} e^{tA_\lambda}x$ exists if $x \in D(A)$

8. Let $E_t x = \lim_{\lambda\to\infty} e^{tA_\lambda}x$ for $x \in D(A)$.

Conclude from #16 that $\lim_{\lambda\to\infty} e^{tA_\lambda}x$ exists

for all $x \in X$.

19 Show $\|E_t x\| \leq e^{-at}\|x\|$ for $t \geq 0$, which establishes (c)

20 Show that $\|E_{s+t}x - E_s E_t x\|$

$$\leq \|(E_{s+t} - e^{(s+t)A_\lambda})x\|$$

$$+ \exp\left(\frac{-as\lambda}{a+\lambda}\right)\|(e^{tA_\lambda} - E_t)x\|$$

$$+ \|(e^{sA_\lambda} - E_s)E_t x\|$$

Conclude that $E_{s+t} = E_s E_t$, which gives (a)

(21) Show that

$$\| e^{tA_\lambda} x - e^{t_0 A_\lambda} x \|$$

$$\leq \int_{t_0}^{t} \| e^{sA_\lambda} A_\lambda x \| ds \leq (t - t_0) A_\lambda x \quad \text{for all } x \in X.$$

Let $x \in D(A)$. Let $\lambda \to \infty$ in the preceding to conclude

$$\| E_t x - E_{t_0} x \| \leq (t - t_0) \| Ax \|$$

Show $E_t x$ is continuous in t for $x \in X$.

(22) From #21 we have

$$e^{tA_\lambda} x - e^{t_0 A_\lambda} x = \int_{t_0}^{t} e^{sA_\lambda} A_\lambda x \, ds$$

(a) If $x \in D(A)$, show that

$$\| \int_{t_0}^{t} e^{sA_\lambda} A_\lambda x \, ds - \int_{t_0}^{t} E_s A x \, ds \|$$

$$\leq \int_{t_0}^{t} e^{-as\lambda/a+\lambda} \| A_\lambda x - Ax \| ds + \int_{t_0}^{t} \| e^{sA_\lambda} Ax - E_s Ax \| ds$$

Now $\| e^{sA_\lambda} (Ax) - E_s Ax \| \to 0$ as $\lambda \to \infty$ by definition of E_s. Show that $\| e^{sA_\lambda} Ax - E_s Ax \| \leq 2 \| Ax \|$ for $s \in [t_0, t]$ and appeal to Dominated Convergence to obtain the second integral tends to 0.

(b) Conclude that

$$E_t x - E_{t_0} x = \int_{t_0}^{t} E_s A x \, ds \qquad \text{if} \quad x \in D(A)$$

Show that

$$\left\| \frac{E_t x - E_{t_0} x}{t - t_0} - E_{t_0} A x \right\| \le \frac{1}{t - t_0} \int_{t_0}^{t} \| E_s A x - E_{t_0} A x \| ds$$

Show that this implies $\left. \frac{d}{dt} E_t x \right|_{t = t_0} = E_{t_0} A x$ for $x \in D(A)$

(c) Assuming $E_t (\lambda - A)^{-1} = (\lambda - A)^{-1} E_t$, $\lambda \ge b$, $t \ge 0$,

show that $A E_t x = (b - (b - A)) E_t x$

$$= (b E_t x) - (b - A) E_t (b - A)^{-1} (b - A) x$$

$$= E_t A x$$

so that $\frac{d}{dt} E_t x = A E_t x$.

23 Establish $E_t (\lambda - A)^{-1} = (\lambda - A)^{-1} E_t$ by completing the following outline:

(a) If $x \in D(A)$, $\lambda \ge b$,

$$A (A - \lambda)^{-1} x = (A - \lambda + \lambda)(\lambda - A)^{-1} x = (A - \lambda)^{-1} A x$$

(b) Let $\mu \ge b$ and let $x \in X$.

Show that $A_\lambda (A-\mu)^{-1} x = \lambda A (A-\lambda)^{-1} (A-\mu)^{-1} x$

$$= (A-\mu)^{-1} A_\lambda x$$

and that

$$A_\lambda^2 (A-\mu)^{-1} x = (A-\mu)^{-1} A_\lambda^2 x$$

Conclude that $\displaystyle\sum_{k=1}^{N} \frac{1}{k!} t^k A_\lambda^k (A-\mu)^{-1} x = (A-\mu)^{-1} \sum_{k=1}^{N} \frac{1}{k!} t^k A_\lambda^k x$

for $\lambda, \mu \geq b$, $x \in X$ and hence

$$e^{tA_\lambda} (\mu - A)^{-1} x = (\mu - A)^{-1} e^{tA_\lambda} x$$

Let $\lambda \to \infty$ to conclude that $E_t (\mu - A)^{-1} = (\mu - A)^{-1} E_t$

(24) Let $\sigma_1(d, m, \beta)$ denote the principal eigenvalue of

$$\nabla (d(x) \nabla \psi) + m(x) \psi = \sigma \psi \quad \text{in } \Omega$$

$$d(x) \nabla \psi \cdot \vec{\eta} + \beta(x) \psi = 0 \quad \text{on } \partial\Omega$$

where $\partial\Omega$ is sufficiently smooth, $d(x) \in C^{1+\alpha}(\bar{\Omega})$ with $d(x) \geq d_0 > 0$, $m(x) \in C(\bar{\Omega})$ and $\beta(x) \in C(\partial\Omega)$ with $\beta(x) \geq 0$. If either (i) $m(x)$ is nonconstant or (ii) $\beta(x) \not\equiv 0$, then $\sigma(d, m, \beta)$ is decreasing in d.

(25) Consider the eigenvalue problem

$$(*) \qquad d\Delta\varphi + r\varphi = \sigma\varphi \quad \text{in } \Omega$$
$$\varphi = 0 \quad \text{on } \partial\Omega$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary and where $d > 0$ and $r \in \mathbb{R}$ are constants.

(a) Show that all eigenvalues are real-valued.

(b) Assume for convenience sake that $r \leq 0$. Show that if $\sigma \neq 0$, then $(*)$ can be re-written in the form

$$(**) \qquad A\varphi = \lambda\varphi$$

where $A : C_0^\alpha(\bar{\Omega}) \to C_0^\alpha(\bar{\Omega})$ is compact and $\lambda = 1/\sigma$. Here $C_0^\alpha(\bar{\Omega})$ denote the Hölder continuous functions of exponent $\alpha \in (0,1)$ which vanish on $\partial\Omega$.

(c) Show that if $r \leq 0$, σ is not an eigenvalue of $(*)$ and 0 is not in the resolvent set of A, usually denoted $\rho(A)$.

(d) Use (c) and the result of problem #12 to conclude that $(*)$ has an infinite sequence of eigenvalues

$$0 > \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k \dots$$

with $\sigma_k \to -\infty$ as $k \to \infty$, if $r \leq 0$.

(e) Explain what happens when $r > 0$.

(26) Repeat #25 if Ψ satisfies

$$d(x)\,\nabla\Psi\cdot\vec{\eta} + \beta(x)\Psi = 0 \quad \text{on } \partial\Omega$$

where $\beta : \partial\Omega \to [0,\infty)$ is continuous and $\beta \not\equiv 0$.

(27) Assume the following result, whose proof can be found in P. Hess, "Periodic-Parabolic Boundary Value Problems and Positivity", Pitman Research Notes in Mathematics Series #247, Longman Scientific and Technical, Harlow, Essex, UK, 1991.

Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary. Let $u : \bar{\Omega} \to \mathbb{R}$ satisfy

$$(*) \qquad \delta(x)\nabla u\cdot\vec{\eta} + \beta(x)u = 0 \qquad \text{on } \partial\Omega$$

where $\delta, \beta \in C^{1+\alpha}(\partial\Omega)$ nonnegative with $\delta(x) + \beta(x) > 0$ for $x \in \partial\Omega$. Suppose that L acting on u satisfying $(*)$ is given by

$$Lu = \nabla\cdot d(x)\nabla u + \vec{b}(x)\cdot\nabla u + c(x)u$$

where $d \in C^{1+\alpha}(\bar{\Omega})$ satisfies $d(x) \geq d_0 > 0$ on $\bar{\Omega}$, $\vec{b} \in [C^{1+\alpha}(\bar{\Omega})]^n$, $c \in C^{\alpha}(\bar{\Omega})$ with $c(x) \leq 0$ on $\bar{\Omega}$.

Suppose that $m : \bar{\Omega} \to \mathbb{R}$ is continuous with $m(x_0) > 0$

(If $\beta(x) \equiv 0$ and $c(x) \equiv 0$, assume $\int_\Omega m(x)\,dx < 0$).

Then the problem

$$[-L - \lambda m(x)]u = f(x) \qquad \text{in } \Omega$$

$$\delta(x)\nabla u \cdot \vec{\eta} + \beta(x)u = 0 \qquad \text{on } \partial\Omega$$

with $f(x) \geq 0$, $f(x_0) > 0$ for some $x_0 \in \bar{\Omega}$, and f continuous, has a unique positive solution if $0 < \lambda < \lambda_1^+(m)$ and no positive solution if $\lambda \geq \lambda_1^+(m)$, where $\lambda_1^+(m)$ denotes the principal positive eigenvalue of the problem

$$-Lu = \lambda m u \qquad \text{in } \Omega$$

$$\delta(x)\nabla u \cdot \vec{\eta} + \beta(x)u = 0 \quad \text{on } \partial\Omega.$$

Prove that for $\lambda > 0$, the principal eigenvalue $\sigma_1(\lambda)$ of

$$L\psi + \lambda m\psi = \sigma\psi \qquad \text{in } \Omega$$

$$\delta(x)\nabla\psi \cdot \vec{\eta} + \beta(x)\psi = 0 \quad \text{on } \partial\Omega$$

with $\psi(\lambda) > 0$ in Ω satisfies

$$\sigma_1(\lambda) < 0 \iff \lambda < \lambda_1^+(m).$$

(28) Consider the diffusive logistic model

$$(*) \quad \begin{aligned} u_t &= \Delta u + (a-u)u && \text{in } \Omega \times (0,\infty) \\ u &= 0 && \text{on } \partial\Omega \times (0,\infty). \end{aligned}$$

Let $a > \lambda_0^1(\Omega)$, the principal eigenvalue of the problem

$$\begin{aligned} -\Delta \phi &= \lambda \phi && \text{in } \Omega \\ \phi &= 0 && \text{on } \partial\Omega \end{aligned}$$

and let $u^*(a)$ denote the minimal positive equilibrium solution of $(*)$. Use Green's Second Identity to prove that $u^*(a)$ is, in fact, the unique positive equilibrium of $(*)$

(29) Suppose u is a positive solution of

$$(*) \quad \begin{aligned} \Delta u + (m(x) - u)u &= 0 && \text{in } \Omega \\ \nabla u \cdot \vec{\eta} &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $m \in C^2(\bar{\Omega})$, $m > 0$ in $\bar{\Omega}$ and m is non-constant. Prove that

$$\int_\Omega m < \int_\Omega u .$$

(30) An important aspect of studying reaction-diffusion-advection models in ecology is having various quantities in the models depend differentiably on other quantities. To this end consider the eigenvalue problem

$$(*) \quad \nabla\cdot d(x)\nabla\psi + \lambda m(x)\psi = \sigma\psi \quad \text{in } \Omega$$
$$\psi = 0 \quad \text{on } \partial\Omega$$

and let $\sigma_1 = \sigma_1(\lambda)$ denote the principal eigenvalue of $(*)$ which denotes the average growth rate over Ω. Complete the following outline of a proof that if $\psi_1(\lambda, x)$ is the corresponding positive eigenfunction normalized by

$$(**) \qquad \int_\Omega \psi_1^2\, dx = 1,$$

then σ_1 is differentiable with respect to λ and

$$\sigma_1'(\lambda) = \int_\Omega m(x)\psi_1^2\, dx$$

(a) Write $(*)-(**)$ as

$$F(\lambda, \psi_1, \sigma_1) = 0$$

where $F : \mathbb{R}\times(C_0^{2+\alpha}(\bar\Omega)\times\mathbb{R}) \to C^\alpha(\bar\Omega)\times\mathbb{R}$ is given by

$$F(\lambda, \psi_1, \sigma_1)$$

$$= \left(\nabla\cdot d(x)\nabla\psi_1 + \lambda m(x)\psi_1 - \sigma_1\psi_1,\ \int_\Omega \psi_1^2 dx - 1\right)$$

(b) Set $X = \mathbb{R}$, $Y = C_0^{2+\alpha}(\bar\Omega)\times\mathbb{R}$, $Z = C^\alpha(\bar\Omega)\times\mathbb{R}$

Assuming, say, that $d \in C^{1+\alpha}(\bar{\Omega})$ and $m \in C^{\alpha}(\bar{\Omega})$, F is a smooth map from $X \times Y$ into Z.
Calculate that

$$F_y(\lambda, \varphi_1, \sigma_1)(\rho, \tau)$$

$$= \left(\nabla \cdot d(x)\nabla\rho + \lambda m(x)\rho - \sigma_1\rho - \tau\varphi_1,\ 2\int_\Omega \varphi_1 \rho\, dx\right)$$

ⓒ To employ the Implicit Function Theorem to establish that σ_1 (and φ_1) are differentiable in λ near $(\lambda, \varphi_1(\lambda), \sigma_1(\lambda))$ boils down to showing that $F_y(\lambda, \varphi(\lambda), \sigma_1(\lambda))$ is invertible.
Show that

$$\nabla \cdot d(x)\nabla\rho + \lambda_0 m(x)\rho - \sigma_1(\lambda_0)\rho - \tau\varphi_1(\lambda_0) = h(x)$$

$$2\int \varphi_1(\lambda_0)\rho\, dx = r$$

is uniquely solvable for arbitrary $h \in C^{\alpha}(\bar{\Omega})$ and $r \in \mathbb{R}$.

ⓓ Differentiate (*) and (**) with respect to λ. Then use integration techniques to obtain the desired formula.